Closing Tues: 12.1,12.2,12.3

Closing Thurs: 12.4(1),12.4(2),12.5(1)

Summary: Vector operations so far

Scalar multiplication

c**v** = "vector parallel to **v** with length scaled by factor of *c*"

Vector Addition

a + **b** = "if **a** and **b** are drawn tail-to-head, then **a** + **b** is the vector that goes from the tail of **a** to the head of **b**"

(resultant/combined force)

12.3 Dot Products *(new)*

If **a** = < a_1, a_2, a_3 > and **b** = < b_1, b_2, b_3 >

Then we define the dot product by:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Note: The dot product gives a number (scalar).

Entry Task: **a** = < 3, 1, 2 >, c = 4

b = - **i** + 6**j** + 5**k**

Compute

1. c**a**
2. unit vector in the direction of **a.**
3. **a** + **b**
4. **a** · **b**

Basic fact list:

- Manipulation facts (*like regular multiplication*):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$\mathrm{c}(\mathbf{a} \cdot \mathbf{b}) = (\mathrm{c}\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathrm{c}\mathbf{b})$$

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = ???$$

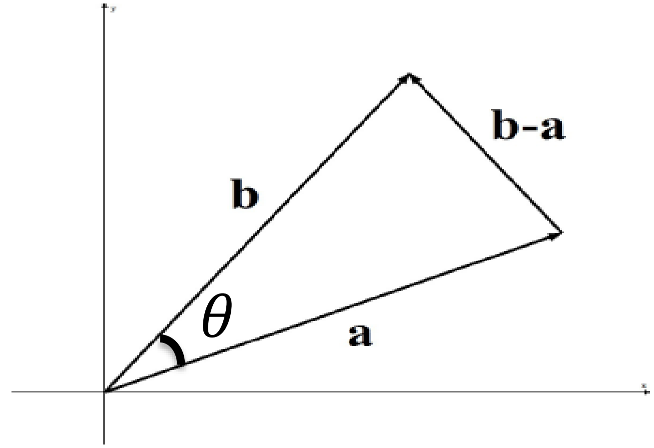
- Helpful fact:

$$\mathbf{a} \cdot \mathbf{a} = \mathrm{a}_1^2 + a_2^2 + a_3^2 = |\boldsymbol{a}|^2$$

The most important fact:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta),$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \leq \theta \leq \pi)$



Proof (not required):

(1) By the Law of Cosines:

$$|\mathbf{b} - \mathbf{a}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos(\theta)$$

(2) The left-hand side expands to

$$\begin{aligned} |\mathbf{b} - \mathbf{a}|^2 &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2 \end{aligned}$$

Subtracting $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$ from both sides of (1) yields:

$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}| \cos(\theta).$$

Divide by -2 to get the result. (QED)

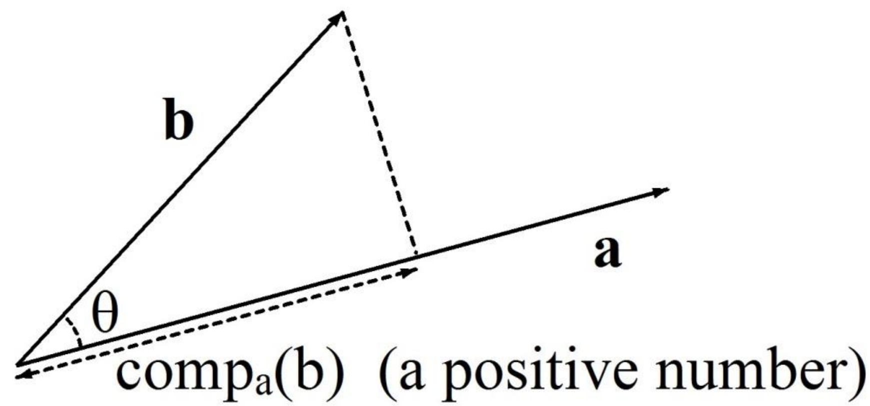
Most important consequence:
If **a** and **b** are orthogonal, then

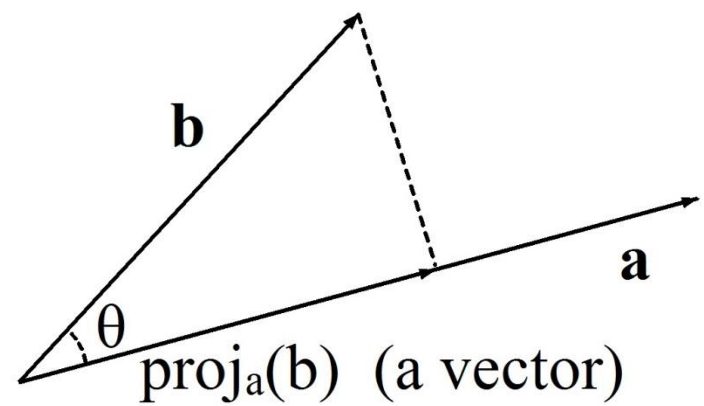
$$\mathbf{a}\cdot\mathbf{b}=0.$$

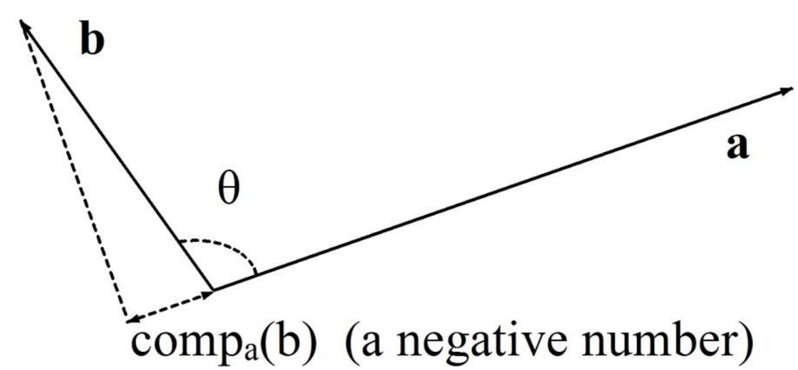
Also: If **a** and **b** are parallel, then
$\mathbf{a}\cdot\mathbf{b}=|\mathrm{a}||\mathrm{b}|$ or $\mathbf{a}\cdot\mathbf{b}=-|\mathrm{a}||\mathrm{b}|$.

Example: Find a vector that is orthogonal to the tangent line to $y=x^3\mathrm{e}^{(2\mathrm{x}-2)}$ at $x=1$.

Projections:







12.4 The Cross Product

We define the <u>cross product</u>, or <u>vector product</u>, for two 3-dimensional vectors,

$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and

$\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$,

by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

$$= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

Ex: $\boldsymbol{a} = \langle 1,2,0 \rangle$ and $\boldsymbol{b} = \langle -1,3,2 \rangle$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$(\quad - \quad)\boldsymbol{i} - (\quad - \quad)\boldsymbol{j} + (\quad - \quad)\boldsymbol{k}$$

You do: $\boldsymbol{a} = \langle 1,3,-1 \rangle$, $\boldsymbol{b} = \langle 2,1,5 \rangle$.

Compute $\boldsymbol{a} \times \boldsymbol{b}$

Most important fact:
The vector $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to *both* **a** and **b**.

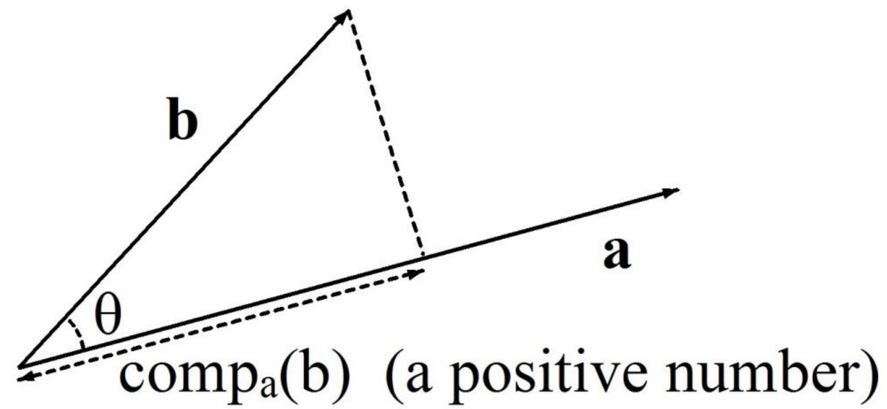
Right-hand rule
If the fingers of the right-hand curl from **a** to **b**, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$:

Through some algebra and using the dot product rule, it can be shown that

$$|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \leq \theta \leq \pi)$



Note: $|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$ is the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$